J. Fluid Mech. (1977), *vol.* 82, *part* 3, *pp.* 469–479
Printed in Great Britain

Nonlinear stability of parallel flows with subcritical Reynolds numbers. Part 2. Stability of pipe Poiseuille flow to finite axisymmetric disturbances

By NOBUTAKE ITOH

National Aerospace Laboratory, Chofu, Tokyo, Japan

(Received 23 November 1976)

A theoretical study is presented of the spatial stability of flow in a circular pipe to small but finite axisymmetric disturbances. The disturbance is represented by a Fourier series with respect to time, and the truncated system of equations for the components up to the second-harmonic wave is derived under a rational assumption concerning the magnitudes of the Fourier components. The solution provides a relation between the damping rate and the amplitude of disturbance. Numerical calculations are carried out for Reynolds numbers R between 500 and 4000 and $\beta R \leqslant 5000$, β being the non-dimensional frequency. The results indicate that the flow is stable to finite disturbances as well as to infinitesimal disturbances for all values of R and βR concerned.

1. Introduction

The stability problem for Poiseuille flow in a circular pipe has been of continuing interest since the classic experiments of Osborne Reynolds (1883), who concluded that the flow would be unstable if the Reynolds number exceeded a certain critical value. Subsequent experiments have shown that the critical Reynolds number varies from about 2000 to 50000 or even more according as the flow at the entrance is disturbed or maintained smooth.

A number of theoretical investigations have been made to determine the critical Reynolds number for this flow. The theory of stability to infinitesimal disturbances, the linear stability theory, was begun with Sexl (1927) and studied definitively by Gill (1965) and Davey & Drazin (1969) for axisymmetric modes of disturbance. Contributions to the harder problem of non-axisymmetric modes have been made by Lessen, Sadler & Liu (1968), Salwen & Grosch (1972) and Garg & Rouleau (1972). All these studies have led to the conclusion that fully developed Poiseuille flow in a pipe is stable to both axisymmetric and non-axisymmetric infinitesimal disturbances at any Reynolds number. On the basis of the concept that a small disturbance could grow near the pipe entrance, where the fully developed parabolic profile has not yet been attained, Tatsumi (1952) studied the stability of the boundary layer in the entrance region to axisymmetric disturbances and found a critical Reynolds number of nearly 10000. This result suggests that the disturbance amplified in the entrance region might reach the fully developed region with an amplitude much larger than the one assumed in the linear stability theory.

It is therefore of great interest to investigate the stability of the fully developed flow to finite amplitude disturbances. Unlike plane Poiseuille flow and Blasius flow, pipe Poiseuille flow has no critical Reynolds number for infinitesimal disturbances as stated above. This fact constituted a great obstacle to the application to the present problem of nonlinear stability theory, as developed by Stuart (1960) and Watson (1960, 1962). For such cases, the method of the false problem was first suggested by Reynolds & Potter (1967, § 4). They allowed the frequency of the fundamental wave to be complex in § 4 of their paper, although they had defined it as a real quantity in the preceding sections. Davey & Nguyen (1971) applied this method to the problem of pipe Poiseuille flow and concluded that the flow became unstable to axisymmetric disturbances if the disturbance amplitude exceeded a certain critical value, the so-called equilibrium amplitude; they calculated the equilibrium amplitude for a range of wavenumbers and Reynolds numbers. Their nonlinear study deals with disturbances which grow or decay with time (temporally dependent disturbances). However, it is preferable for comparison with experimental results to consider disturbances which grow or decay with downstream distance (spatially dependent disturbances).

In the work reported here the stability of pipe Poiseuille flow to spatially dependent axisymmetric disturbances with small but finite amplitude is investigated with the aid of the asymptotic theory valid for subcritical flows presented in part 1 (Itoh 1977*a*), together with the method of analysis for the spatial development of finite amplitude disturbances given in the author's previous papers (Itoh 1974*b*, *c*), in which the method of Watson (1962) was modified to examine the two-dimensional cases of plane Poiseuille flow and of the boundary-layer flow on a flat plate. The numerical method developed by the author (Itoh 1974*a*) is used for integrating ordinary differential equations of fourth order. It is shown in § 3 that the present method leads to a result opposite to that obtained by Davey & Nguyen on the basis of the formulation of Reynolds & Potter; that is, there is no equilibrium amplitude in the case of pipe Poiseuille flow. A lucid explanation for these contradictory results is given in § 4.

2. Linear theory

Only axisymmetric motion of an incompressible fluid in a circular pipe is considered. The velocity profile of the basic flow is parabolic with maximum velocity U_0 on the centre-line. All quantities are made non-dimensional with the pipe radius a, the velocity U_0 and the reference time a/U_0. The Reynolds number of the flow is $R = U_0 a/\nu$, where ν is the kinematic viscosity. Let (x, r) be co-ordinates in the axial and radial directions, respectively, and (u, v) the corresponding velocity components. Since the quantities are independent of the azimuthal angle, the velocity field can be expressed in terms of a stream function Ψ as

$$u = r^{-1}\,\partial\Psi/\partial r, \quad v = -r^{-1}\,\partial\Psi/\partial x. \tag{2.1}$$

In order to investigate the behaviour of a small disturbance superimposed on the basic flow, the stream function is divided into two terms as

$$\Psi = \Psi_0(r) + \psi(r, x, t), \tag{2.2}$$

where $\Psi_0(r) = \frac{1}{2}r^2 - \frac{1}{4}r^4$ represents the basic flow and $\psi(r, x, t)$ the axisymmetric

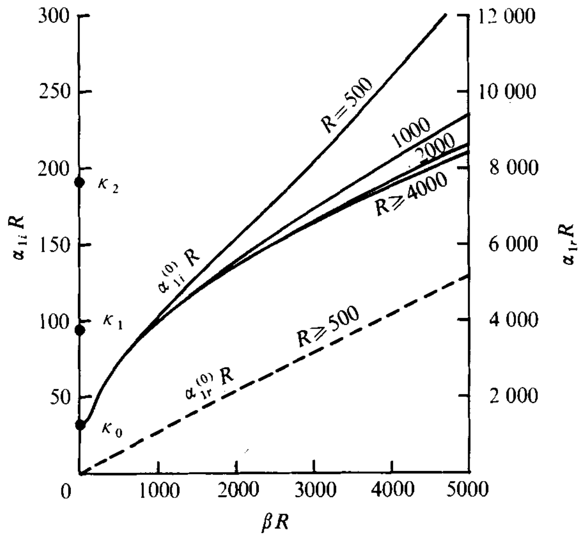

FIGURE 1. The variation of the least stable eigenvalue with the non-dimensional frequency. ●, the first three eigenvalues for $\beta = 0$ and $R \geqslant 500$.

disturbance. Substitution of (2.2) into the Navier–Stokes equation leads to the equation for the function $\psi(r, x, t)$:

$$\left\{\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2} - R\left(\frac{\partial}{\partial t} + \frac{1}{r}\frac{d\Psi_0}{dr}\frac{\partial}{\partial x}\right)\right\}\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}\right)\psi$$
$$= \frac{R}{r}\left\{\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\left(\frac{\partial}{\partial r} - \frac{2}{r}\right)\right\}\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}\right)\psi. \tag{2.3}$$

The boundary conditions to be imposed are that the disturbance velocity should be axisymmetric, bounded on the centre-line and vanish at the wall. These are

$$\left.\begin{aligned} &r^{-1}\,\partial\psi/\partial x = 0, \quad r^{-1}\,\partial\psi/\partial r < \infty \quad \text{when } r \to 0 \\ \text{and} \quad &\partial\psi/\partial x = \partial\psi/\partial r = 0 \quad \text{at} \quad r = 1. \end{aligned}\right\} \tag{2.4}$$

In the linear stability theory disturbances are assumed to be infinitesimal and the coupling terms on the right-hand side of (2.3) are neglected. The resultant linear equation admits a solution of the form

$$\psi(r, x, t) = A_1\,\phi_1(r)\exp\{i(\alpha_1 x - \beta t)\}, \tag{2.5}$$

where α_1 is a complex wavenumber, β a real frequency, A_1 an arbitrary constant and the function $\phi_1(r)$ is made definite by imposing the normalization condition

$$\phi_1''(0) = 1, \tag{2.6}$$

the primes denoting differentiation with respect to r. Then the equation

$$\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - \alpha_1^2 - iR\left(-\beta + \frac{\alpha_1}{r}\frac{d\Psi_0}{dr}\right)\right\}\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - \alpha_1^2\right)\phi_1(r) = 0, \tag{2.7}$$

with the boundary conditions (2.4), provides an eigenvalue problem determining an infinite sequence of eigenvalues $\alpha_1^{(n)}$ $(n = 0, 1, 2, \ldots)$ as functions of β and R. The imaginary part of each eigenvalue represents the damping rate of the corresponding mode.

This eigenvalue problem has been solved by Gill (1965), Davey & Drazin (1969) and Garg & Rouleau (1972). From these studies, it is found that the imaginary parts of all the eigenvalues are always positive, indicating that all infinitesimal disturbances decay with downstream distance in pipe Poiseuille flow. Since results are given in those papers only for high Reynolds numbers, the least stable eigenvalue $\alpha_1^{(0)}$ has been recalculated for a wide range $R \geqslant 500$ and $\beta R \leqslant 5000$ and is shown in figure 1, where the curves of the real part $\alpha_{1r}^{(0)} R$ are indistinguishable for $R \geqslant 500$. The first three purely imaginary eigenvalues for $\beta = 0$ and $R \geqslant 500$ ($\alpha_1^{(n)} = i\kappa_n$ for $n = 0, 1, 2$), which are eigenvalues of the mean-flow distortion, are also given in figure 1. The results are in good agreement with those of Davey & Drazin (1969) and others.

3. Nonlinear theory

In order to solve the nonlinear disturbance equation (2.3), we expand the function ψ in a Fourier series with respect to time:

$$\psi(r,x,t) = \psi_0(r,x) + \psi_1(r,x)\, e^{-i\beta t} + \tilde{\psi}_1(r,x)\, e^{i\beta t} + \psi_2(r,x)\, e^{-2i\beta t} + \tilde{\psi}_2(r,x)\, e^{2i\beta t} + \text{higher-order terms}, \tag{3.1}$$

where a tilde denotes a complex conjugate. Substituting (3.1) into (2.3) and equating the Fourier components, we obtain an infinite set of equations. Following Stuart (1960) and Watson (1960, 1962), we assume that the amplitude $|A_1|$ of the fundamental wave is sufficiently small, and that $\psi_0 = O(|A_1|^2)$ and $\psi_n = O(|A_1|^n)$ for $n \geqslant 1$. Then the higher-order terms in (3.1) can be neglected, and the equations for the fundamental disturbance ψ_1, the mean-flow distortion ψ_0 and the second-harmonic disturbance ψ_2 are obtained in the form

$$L_1[\psi_1] = M[\psi_0, \psi_1] + M[\psi_2, \tilde{\psi}_1], \tag{3.2a}$$

$$L_0[\psi_0] = M[\psi_1, \tilde{\psi}_1], \quad L_2[\psi_2] = \tfrac{1}{2} M[\psi_1, \psi_1], \tag{3.2b,c}$$

where the operators L_k $(k = 0, 1, 2)$ and M are defined by

$$\left.\begin{aligned} L_k[\psi] &= \left\{\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2} - R\left(-ik\beta + \frac{1}{r}\frac{d\Psi_0}{dr}\frac{\partial}{\partial x}\right)\right\}\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}\right)\psi, \\ M[\psi, \phi] &= \frac{R}{r}\left[\left\{\frac{\partial\psi}{\partial r}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\left(\frac{\partial}{\partial r} - \frac{2}{r}\right)\right\}\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}\right)\phi \right. \\ &\qquad \left. + \left\{\frac{\partial\phi}{\partial r}\frac{\partial}{\partial x} - \frac{\partial\phi}{\partial x}\left(\frac{\partial}{\partial r} - \frac{2}{r}\right)\right\}\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}\right)\psi\right]. \end{aligned}\right\} \tag{3.3}$$

The solution of this set of equations is considered to describe the behaviour of finite disturbances to a fairly good approximation when their amplitudes are not very large.

In order to solve (3.2), we consider a fundamental disturbance of the form

$$\psi_1(r,x) = A_1\, \phi_1(r) \exp(i\alpha_1 x), \tag{3.4a}$$

with

$$\phi_1(r) = \phi_1^{(0)}(r) + |A_1|^2 f(r) + O(|A_1|^4), \tag{3.4b}$$

$$\alpha_1 = \alpha_1^{(0)} + \lambda |A_1|^2 + O(|A_1|^4), \tag{3.4c}$$

where the constant A_1 denotes a complex amplitude defined at the origin $x = 0$, which may be arbitrarily chosen along the x axis, and $\alpha_1^{(0)}$ and $\phi_1^{(0)}$ are the least stable eigenvalue and the corresponding eigenfunction of the linear problem. Substituting the first approximation to the fundamental $\psi_1(r,x) = A_1\phi_1^{(0)}(r)\exp(i\alpha_1^{(0)}x)$ into the nonlinear terms of (3.2b, c), we solve the resultant inhomogeneous equations for the mean-flow distortion and second-harmonic disturbance. Then the solutions together with the first approximation to the fundamental are substituted into the right-hand side of (3.2a) to obtain the correction terms $f(r)$ and λ in (3.4).

The equations for the mean-flow distortion and second-harmonic disturbance have forcing terms of the form

$$\left.\begin{aligned} M[\psi_1,\tilde{\psi}_1] &= |A_1|^2 h_0(r)\exp(-2\alpha_{1i}^{(0)}x)+O(|A_1|^4),\\ \tfrac{1}{2}M[\psi_1,\psi_1] &= A_1^2 h_2(r)\exp(2i\alpha_1^{(0)}x)+O(|A_1|^4),\end{aligned}\right\} \tag{3.5}$$

respectively, where

$$\left.\begin{aligned} h_0(r) = \frac{iR}{r}\Bigg[&\left\{\alpha_1^{(0)}\frac{d\tilde{\phi}_1^{(0)}}{dr}+\tilde{\alpha}_1^{(0)}\tilde{\phi}_1^{(0)}\left(\frac{d}{dr}-\frac{2}{r}\right)\right\}\left\{\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}-(\alpha_1^{(0)})^2\right\}\phi_1^{(0)}\\ &-\left\{\tilde{\alpha}_1^{(0)}\frac{d\phi_1^{(0)}}{dr}+\alpha_1^{(0)}\phi_1^{(0)}\left(\frac{d}{dr}-\frac{2}{r}\right)\right\}\left\{\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}-(\tilde{\alpha}_1^{(0)})^2\right\}\tilde{\phi}_1^{(0)}\Bigg],\\ h_2(r) = \frac{i\alpha_1^{(0)}R}{r}&\left\{\frac{d\phi_1^{(0)}}{dr}-\phi_1^{(0)}\left(\frac{d}{dr}-\frac{2}{r}\right)\right\}\left\{\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}-(\alpha_1^{(0)})^2\right\}\phi_1^{(0)}.\end{aligned}\right\} \tag{3.6}$$

A formal solution of these equations can be obtained by following the formulation of Watson (1962), who put

$$\psi_0 = |A_1|^2 G_0(r)\exp(-2\alpha_{1i}^{(0)}x)+O(|A_1|^4), \tag{3.7}$$

$$\psi_2 = A_1^2 G_2(r)\exp(2i\alpha_1^{(0)}x)+O(|A_1|^4). \tag{3.8}$$

On the other hand, Davey & Nguyen (1971), following the method of Reynolds & Potter (1967, § 4), ignored the term $\exp(-2\alpha_{1i}^{(0)}x)$ in the forcing term of the mean-flow equation and put the solution in the equilibrium form $\psi_0 = |A_1|^2 g_0(r)$, although they used the non-equilibrium form (3.8) for the solution of the second-harmonic equation. However, both these methods appear to be inadequate in the case of pipe Poiseuille flow, because the damping rate $\alpha_{1i}^{(0)}$ obtained from linear theory is not of smaller order than the damping rates of the eigensolutions of the mean-flow and second-harmonic equations. This fact is shown in figure 1; for instance, when $R = 1000$ and $\beta = 2{\cdot}0$, we have $\alpha_{1i}^{(0)} = 0{\cdot}14$, while the damping rates of the mean-flow distortion, for which $\beta = 0$, and of the second harmonic, for which $\beta = 4{\cdot}0$, are 0·03 and 0·21, respectively, both of which are less than $2\alpha_{1i}^{(0)}$. Therefore the eigensolution components cannot be ignored in solutions of the second-harmonic equation or the mean-flow equation. Further discussions of this point are given in § 4.

In order to take into account the eigensolution components mentioned above, we follow the approach presented in part 1 (Itoh 1977a). The main point of that approach is to display the behaviour of the disturbance amplitude in an infinite-dimensional phase space introduced by the method of eigenfunction expansion, which is considered to be applicable to the present problem of spatially dependent disturbances. The phase-space consideration indicates that the quantity λ, which, together with the linear damping rate $\alpha_{1i}^{(0)}$, determines the pattern of development of weakly nonlinear disturbances, should be calculated by using mean-flow and second-harmonic components of the equilibrium form

$$\psi_0 = |A_1|^2 g_0(r) + O(|A_1|^4) \tag{3.9}$$

and

$$\psi_2 = A_1^2 g_2(r) \exp(2i\alpha_{1r}^{(0)} x) + O(|A_1|^4), \tag{3.10}$$

respectively, where the functions $g_0(r)$ and $g_2(r)$ are the solutions of the equations

$$\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right) g_0(r) = h_0(r), \tag{3.11}$$

$$\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - 4(\alpha_{1r}^{(0)})^2 - 2iR\left(-\beta + \frac{\alpha_{1r}^{(0)}}{r}\frac{d\Psi_0}{dr}\right)\right\}\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - 4(\alpha_{1r}^{(0)})^2\right\} g_2(r) = h_2(r) \tag{3.12}$$

under the boundary conditions

$$\lim_{r\to 0} r^{-1} g_k(r) = 0, \quad \lim_{r\to 0} r^{-1} g_k'(r) < \infty, \quad g_k(1) = g_k'(1) = 0 \quad (k = 0, 2). \tag{3.13}$$

Substituting (3.9) and (3.10) together with the first approximation to ψ_1 into the right-hand side of (3.2 a), and (3.4) into the left-hand side, then equating the coefficients of $A_1|A_1|^2$ on both sides, we obtain the equation determining the correction terms $f(r)$ and λ in (3.4) as

$$L_{10}[f(r)] = -\lambda L_{11}[\phi_1^{(0)}(r)] + H_0(r) + H_2(r), \tag{3.14}$$

where the operators L_{10} and L_{11} and the functions $H_0(r)$ and $H_2(r)$ are defined by

$$\left.\begin{aligned}
L_{10} &= \left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - (\alpha_1^{(0)})^2 - iR\left(-\beta + \frac{\alpha_1^{(0)}}{r}\frac{d\Psi_0}{dr}\right)\right\}\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - (\alpha_1^{(0)})^2\right\},\\
L_{11} &= \frac{\partial L_{10}}{\partial \alpha_1^{(0)}} = -\left(4\alpha_1^{(0)} - iR\frac{1}{r}\frac{d\Psi_0}{dr}\right)\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - (\alpha_1^{(0)})^2\right\} + 2i\alpha_1^{(0)} R\left(-\beta + \frac{\alpha_1^{(0)}}{r}\frac{d\Psi_0}{dr}\right),\\
H_0(r) &= \frac{i\alpha_1^{(0)} R}{r}\left[\frac{dg_0}{dr}\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - (\alpha_1^{(0)})^2\right\}\phi_1^{(0)} - \phi_1^{(0)}\left(\frac{d}{dr} - \frac{2}{r}\right)\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right) g_0\right],\\
H_2(r) &= \frac{iR}{r}\left[\left\{2\alpha_{1r}^{(0)}\frac{d\tilde{\phi}_1^{(0)}}{dr} + \tilde{\alpha}_1^{(0)}\tilde{\phi}_1^{(0)}\left(\frac{d}{dr} - \frac{2}{r}\right)\right\}\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - 4(\alpha_{1r}^{(0)})^2\right\} g_2\right.\\
&\qquad \left. - \left\{\tilde{\alpha}_1^{(0)}\frac{dg_2}{dr} + 2\alpha_{1r}^{(0)} g_2\left(\frac{d}{dr} - \frac{2}{r}\right)\right\}\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - (\tilde{\alpha}_1^{(0)})^2\right\}\tilde{\phi}_1^{(0)}\right].
\end{aligned}\right\} \tag{3.15}$$

Since the above equation has the same operator on the left-hand side as the linear equation (2.7), a solution exists only when the right-hand side satisfies a solvability condition. If we introduce the adjoint eigenfunction $\Phi(r)$ associated with the linear equation (2.7), the solvability condition is written in the form

$$\int_0^1 \Phi(r)\{-\lambda L_{11}[\phi_1^{(0)}(r)] + H_0(r) + H_2(r)\}\, dr = 0, \tag{3.16}$$

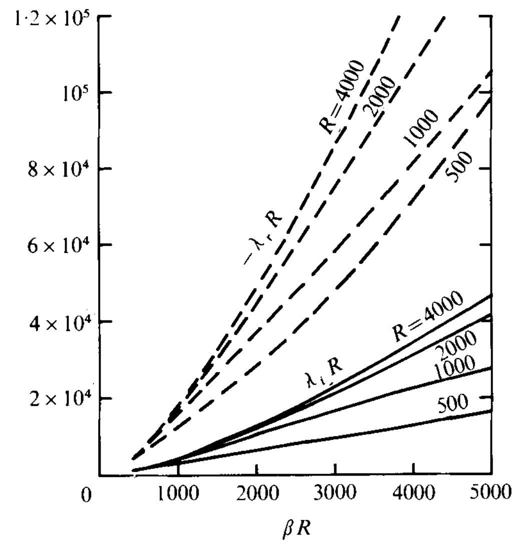

FIGURE 2. The variation of $\lambda_r R$ and $\lambda_i R$ with βR.

which determines the value of λ. The function $f(r)$ is determined uniquely by imposing the boundary conditions

$$\lim_{r\to 0} r^{-1}f(r) = 0, \quad \lim_{r\to 0} r^{-1}f'(r) < \infty, \quad f(1) = f'(1) = 0 \tag{3.17}$$

and an additional condition $$f''(0) = 0, \tag{3.18}$$

which originates from the fact that ψ_1/A_1 should satisfy the normalization condition (2.6). Equation (3.16) indicates that the coefficient λ can be divided into two parts as

$$\lambda = \lambda_0 + \lambda_2, \quad \lambda_0 = \frac{\int_0^1 \Phi(r)\, H_0(r)\, dr}{\int_0^1 \Phi(r)\, L_{11}[\phi_1^{(0)}(r)]\, dr}, \quad \lambda_2 = \frac{\int_0^1 \Phi(r)\, H_2(r)\, dr}{\int_0^1 \Phi(r)\, L_{11}[\phi_1^{(0)}(r)]\, dr}. \tag{3.19}$$

If the higher-order terms of (3.1) are taken into account, more terms of order $|A_1|^{2n}$ $(n \geqslant 2)$ on the right-hand sides of (3.4*b, c*) are determined in a similar way. The damping rate of finite amplitude disturbances is generally written in the form

$$\alpha_{1i} = \alpha_{1i}^{(0)} + \lambda_i |A_1|^2 + O(|A_1|^4). \tag{3.20}$$

The values $|A_1| = |A_1|_e$ for which $\alpha_{1i} = 0$ are called the equilibrium amplitudes. If an equilibrium amplitude exists in the range of small amplitudes for which the term of order $|A_1|^2$ in (3.20) will be the dominant nonlinear term, then the equilibrium amplitude is approximately given by

$$|A_1|^2 = -\alpha_{1i}^{(0)}/\lambda_i. \tag{3.21}$$

The results of the linear theory have shown that the damping rate $\alpha_{1i}^{(0)}$ of infinitesimal disturbances is positive for the whole range of frequencies and Reynolds numbers. Therefore the small equilibrium amplitude exists only when λ_i is negative.

Numerical calculations have been carried out in order to determine the values of

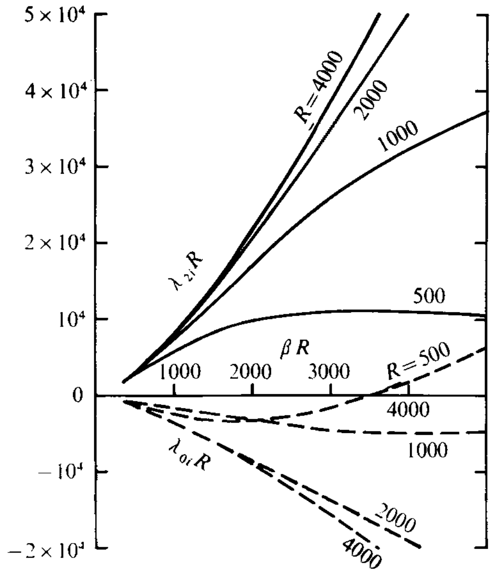

FIGURE 3. The variation of $\lambda_{0i}R$ and $\lambda_{2i}R$ with βR.

λ for $R = 500$, 1000, 2000 and 4000 and $\beta R \leqslant 5000$. The results are shown in figure 2, where the real and imaginary parts of λR are plotted against βR for each value of the Reynolds number. It is seen that λ_i is always positive, indicating that no equilibrium state exists for small amplitudes. A small but finite disturbance thus turns out to decay faster than the corresponding infinitesimal disturbance. We note from (3.19) that the coefficient λ_i is made up of two terms representing the effect of the mean-flow distortion (λ_{0i}) and the effect of the second-harmonic disturbance (λ_{2i}) respectively. The variations of $\lambda_{0i}R$ and $\lambda_{2i}R$ with βR are presented in figure 3. The effect of the second-harmonic disturbance is always to make the flow stable and is much larger than the effect of the mean-flow distortion, which makes the flow unstable in some range of βR.

4. Discussion

The present analysis leads to the result that no equilibrium state can exist for spatially dependent disturbances, while Davey & Nguyen (1971) have found the equilibrium amplitude of temporally dependent disturbances for a range of wave-numbers and Reynolds numbers. It is very important to inquire into the reason for the contradiction between these two studies.

In the paper of Davey & Nguyen, disturbances are temporally growing or decaying waves, so that α_1 is real and β is complex, the imaginary part β_i representing the amplification rate. The linear problem determines eigenvalues $\beta^{(n)}$ ($n = 0, 1, 2, \ldots$) as functions of α_1 and R, $\beta^{(0)}$ corresponding to the least stable one. Davey & Nguyen assumed an equilibrium state when they derived the equation for the mean-flow distortion. The equation was written in the form similar to (3.11), although the boundary conditions were slightly different from (3.12) because the pressure gradient was supposed to be maintained at a constant value in their formulation. On the other hand, however, the equation for the second-harmonic disturbance was considered without making the assumption of an equilibrium state. As a matter of fact, the complex eigenvalue $\beta^{(0)}$ (instead of $\beta_r^{(0)}$) was used as the parameter β appearing in the equation corresponding to (3.12). This formulation brings about the existence of an equilibrium amplitude; for example, when $\alpha_1 = 6{\cdot}2$ and $R = 500$, numerical calculations by the present author show that

$$\beta^{(0)} = 5{\cdot}88504 - 0{\cdot}391840i, \quad \lambda = -3{\cdot}7023 + 35{\cdot}878i$$

and thus the equilibrium amplitude is given by $|A_1| = (-\beta_i^{(0)}/\lambda_i)^{\frac{1}{2}} = 0{\cdot}10451$. These values coincide with those given by Davey & Nguyen (1971, p. 710) except for the difference in the notation.

If the present approach were applied to the temporal problem with the same wavenumber and Reynolds number, we should obtain a negative value of $-\beta_i^{(0)}/\lambda_i$ with $\lambda = 65{\cdot}500 - 20{\cdot}266i$, indicating the absence of an equilibrium amplitude, which is in agreement with the result for the spatial problem given in § 3. It is seen therefore that the only difference between the present approach and that of Davey & Nguyen is in the treatment of the amplification rate of the second-harmonic disturbance. If the complex eigenvalue $2\alpha_1^{(0)}$ had been used in (3.12) instead of $2\alpha_{1r}^{(0)}$, the sign of λ_i would have been opposite to that obtained in § 3. In order to confirm this conjecture, let us write α_2 instead of $2\alpha_{1r}^{(0)}$ for the generalized expression for the complex wavenumber of the second-harmonic disturbance. Then (3.12) may be written in the form

$$\left\{\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - \alpha_2^2 - iR\left(-2\beta + \frac{\alpha_2}{r}\frac{d\Upsilon_0}{dr}\right)\right\}\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - \alpha_2^2\right) g_2(r) = h_2(r). \tag{4.1}$$

It is easily found that the reversal in the sign of λ_i, as mentioned above, arises from the singularity of this equation at $\alpha_2 = \alpha_2^{(0)}$, $\alpha_2^{(0)}$ being the least stable eigenvalue of the corresponding homogeneous equation. The eigenfunction expansion of the particular solution of (4.1) reveals that the solution $g_2(r)$ becomes infinite when the parameter α_2 coincides with one of the eigenvalues. In particular for the value of α_2 in the vicinity of the least stable eigenvalue $\alpha_2^{(0)}$, the solution of (4.1) can be written in the form

$$g_2(r) = \frac{\sigma}{\alpha_2 - \alpha_2^{(0)}} g_2^*(r), \tag{4.2}$$

where σ is a constant and $g_2^*(r)$ a normalized function. The signs of the real and imaginary parts of $g_2(r)$ are reversed according as $\alpha_2 = 2\alpha_{1r}^{(0)}$ or $\alpha_2 = 2\alpha_1^{(0)}$, because of the fact that $\alpha_{2r}^{(0)} \doteqdot 2\alpha_{1r}^{(0)}$ as shown in figure 1. This property of the second-harmonic solution furnishes the major reason for the contradiction between the present results and those of Davey & Nguyen.

It is obvious from the phase-space consideration given in part 1 that the judgement as to whether weak nonlinearity contributes to growth or decay of disturbances should be made with the Landau constant determined by the use of the equilibrium solutions for the mean-flow distortion and the second-harmonic disturbance. That is, the parameter α_2 in (4.1) should be taken to be $2\alpha_{1r}^{(0)}$. The Landau constant λ_i obtained in this way is negative as shown in figure 2, so that nonlinearity, at least when it is weak, accelerates the damping of disturbances in the case of pipe Poiseuille flow.

Here a remark should be made about the formulation of the false-problem method

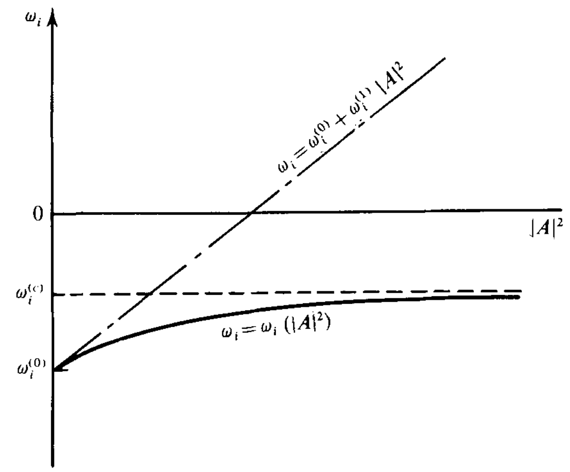

FIGURE 4. Schematic sketch of the relation between ω_i and $|A|^2$ in the false problem.

of Reynolds & Potter (1967, § 4), on which the analysis of Davey & Nguyen is based. In the false problem the complex frequency ω is expanded in powers of the disturbance amplitude $|A|$ and the solutions of the true problem are extracted by selecting the values of $|A|$ which make ω real. This method requires first the convergence of the series

$$\omega_i(|A|^2) = \sum_{n=0}^{\infty} \omega_i^{(n)}|A|^{2n} \tag{4.3}$$

in a range of small amplitudes. Next, it is necessary that there is at least one root of the equation $\omega_i(|A|^2) = 0$ inside the domain of convergence. This means that the curve $\omega_i = \omega_i(|A|^2)$ in the $|A|^2$, ω_i plane must intersect the $|A|^2$ axis at a point in a range of small $|A|^2$. In the case of pipe Poiseuille flow, however, the comparison between the two numerical results given in the second and third paragraphs of this section suggests that a singularity similar to the one in (4.2) exists also in the temporal problem; that is, there is a singular point $\omega_i^{[c]} = \frac{1}{2}\omega_{2i}^{(0)}$ between $\omega_i = \omega_i^{(0)}$ and $\omega_i = 0$ as shown in figure 4, where $\omega_i^{(0)}$ and $\omega_{2i}^{(0)}$ are the amplification rates of the eigensolutions of the fundamental and second-harmonic equations, respectively. This fact indicates that the solution curve $\omega_i = \omega_i(|A|^2)$ of the false problem has no intersection with the $|A|^2$ axis for any small value of $|A|^2$, although the approximation by the first two terms $\omega_i = \omega_i^{(0)} + \omega_i^{(1)}|A|^2$ is supposed to give an equilibrium point. Thus we are led to the conclusion that the false-problem method of Reynolds & Potter cannot be applied to pipe Poiseuille flow.

5. Conclusion

The stability of pipe Poiseuille flow to axisymmetric disturbances has been examined by application of the asymptotic theory which is developed in part 1, which is guaranteed to be valid for subcritical flows. Numerical results for R up to 4000 and βR up to 5000 indicate that this flow is spatially stable to small but finite disturbances as well as to infinitesimal disturbances, provided that the disturbance is sufficiently small and the nth component in the Fourier expansion may be assumed to be of order $\epsilon^{|n-1|+1}$, ϵ being the magnitude of the fundamental. The mean-flow distortion has a tendency to make the flow unstable for some range of R and βR, but the effect is small compared with the stabilizing effect of the second-harmonic disturbance, resulting in stability of the flow for the whole range of Reynolds numbers and wave-numbers concerned.

The above conclusion apparently contradicts the result of Davey & Nguyen (1971) that the flow can be unstable if the disturbance amplitude exceeds a certain value. It is pointed out, however, that the false-problem method of Reynolds & Potter (1967, § 4), on which the formulation of Davey & Nguyen is based, is not applicable to the problem of pipe Poiseuille flow.

The present analysis is unsuccessful in explaining the experimentally observed instability of pipe flow. Seeing that the disturbance considered is limited to the least stable mode of axisymmetric disturbance, the cause of instability is most likely to be found in other modes of disturbance, such as the first azimuthal mode, which is now believed to be the least stable mode among all axisymmetric and non-axisymmetric disturbances (Garg & Rouleau 1972). Also, there seems to be the possibility of instability due to nonlinear interactions between different modes, in view of the strong instability arising from the interaction between two-dimensional and three-dimensional disturbances in plane Poiseuille flow (Itoh 1977 *b*).

The author is greatly indebted to Professor Itiro Tani for his valuable advice and constant encouragement. He also wishes to thank Professor J. T. Stuart for many helpful discussions during the author's visit to the Imperial College of Science and Technology, London.

REFERENCES

DAVEY, A. & DRAZIN, P. G. 1969 *J. Fluid Mech.* **36**, 209.
DAVEY, A. & NGUYEN, H. P. F. 1971 *J. Fluid Mech.* **45**, 701.
GARG, V. K. & ROULEAU, W. T. 1972 *J. Fluid Mech.* **54**, 113.
GILL, A. E. 1965 *J. Fluid Mech.* **21**, 145.
ITOH, N. 1974 *a* *Trans. Japan Soc. Aero. Space Sci.* **17**, 65.
ITOH, N. 1974 *b* *Trans. Japan Soc. Aero. Space Sci.* **17**, 160.
ITOH, N. 1974 *c* *Trans. Japan Soc. Aero. Space Sci.* **17**, 175.
ITOH, N. 1977 *a* *J. Fluid Mech.* **82**, 455.
ITOH, N. 1977 *b* Three-dimensional growth of finite wave disturbances in plane Poiseuille flow. To appear.
LESSEN, M., SADLER, S. G. & LIU, T. Y. 1968 *Phys. Fluids* **11**, 1404.
REYNOLDS, O. 1883 *Phil. Trans. Roy. Soc.* **174**, 935.
REYNOLDS, W. C. & POTTER, M. C. 1967 *J. Fluid Mech.* **27**, 465.
SALWEN, H. & GROSCH, C. E. 1972 *J. Fluid Mech.* **54**, 93.
SEXL, T. 1927 *Ann. Phys.* **83**, 835.
STUART, J. T. 1960 *J. Fluid Mech.* **9**, 353.
TATSUMI, T. 1952 *J. Japan Phys. Soc.* **7**, 489.
WATSON, J. 1960 *J. Fluid Mech.* **9**, 371.
WATSON, J. 1962 *J. Fluid Mech.* **14**, 211.